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A NOTE ON THE MEAN VALUE OF RANDOM DETERMINANTS

Richard Bellman

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary

~~In this paper we present~~ an explicit expression for the moments of a random determinant is presented.

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A Note on the Mean Value of Random Determinants

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1. Introduction. In a recent paper, [ ], Nyquist, Rice and Riordan discussed the problem of determining the expected values of powers of a random determinant. Here a random determinant, D_n, is defined to be

$$D_n = |x_{ij}|, \quad i,j, = 1,2,\ldots,n,$$

where the x_{ij} are independent random variables.

The purpose of the present note is to give an explicit representation for $E(D_n^k)$ in terms of the characteristic functions of the x_{ij}. These need not be identical.

At the moment we are merely interested in presenting an expression which will yield a systematic technique for obtaining the moments numerically. In a subsequent paper devoted to various theoretical aspects such as asymptotic behavior we shall discuss the problem in greater detail. For the case of identical distributions, the problem is closely connected with the study of invariants of the symmetric group. The operator we employ below is related to the operator of Capelli discussed in Weyl's book on the classical groups.

2. A Useful Operator.

Let us consider the operator θ_n defined as

$$\theta_n = |\partial/\partial z_{kl}|, \quad k,l = 1,2,\ldots,n, \tag{2.1}$$

where the z_{kl} are independent variables. Thus

$$\theta_1 = \partial / \partial z_{11}$$

$$\theta_2 = \frac{\partial}{\partial z_{11}} \frac{\partial}{\partial z_{22}} - \frac{\partial}{\partial z_{12}} \frac{\partial}{\partial z_{21}}, \tag{2.2}$$

and so on.

Let X represent the matrix (x_{kl}) and Z the matrix (z_{kl}). Then we have

$$e^{\,itr(XZ^T)} = e^{i \sum_{k,l} x_{kl} z_{kl}} \tag{2.3}$$

The basic identity we shall employ below is

$$\theta^k \left[e^{itr(XZ^T)} \right] = i^{nk} D_n^k \, e^{itr(XZ^T)}, \tag{2.4}$$

for $k = 1, 2, \ldots$.*

3. $E(D_n^k)$.

Taking the expected value of both sides in (2.4), we obtain the result

$$\theta_n^k \left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl}) \right] = i^{nk} E(D_n^k \, e^{i \, tr(XZ^T)}), \tag{3.1}$$

where

$$\phi_{kl}(z) = \int_{-\infty}^{\infty} e^{ixz} dG_{kl}(x), \tag{3.2}$$

is the characteristic function of the random variable x_{kl}.

Setting $z_{kl} = 0$, we obtain the result

$$i^{nk} E(D_n^k) = \theta_n^k \left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl})_{z_{kl} = 0} \right] \tag{3.3}$$

* this is a well-known device in the theory of matric automorphic functions.

4. Identical Distributions.

If the variables are identically distributed and symmetric about zero, we may write

$$\phi(z_{kl}) = e^{a_1 z_{kl}^2 + a_2 z_{kl}^4 + \cdots} \tag{4.1}$$

obtaining as a consequence in place of (3.3) the result

$$i^{nk} E(D_n^k) = \theta_n^k \left[e^{a_1 \sum_{k,l} z_{kl}^2 + a_2 \sum_{k,l} z_{kl}^4 + \cdots} \right]_{z_{kl} = 0} \tag{4.2}$$

From this representation the value of $E(D_n^k)$ may be obtained by retaining in the above expression only the terms that yield a non-zero value after z_{kl} has been set equal to zero.

A particularly interesting case is that where $x_{kl} = \pm 1$ with equal probability. Then

$$i^{nk} E(D_n^k) = \theta_n^k \left[\prod_{k,l=1}^{n} (2\cos z_{kl}) \right]_{z_{kl} = 0} \tag{4.3}$$

Bibliography

1. H. Nyquist, S.O. Rice, and J. Riordan, The Distribution of Random Determinants, Quarterly of Applied Mathematics, Vol XII (1954), pp 97-104.